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A Pseudo-spectral Method and Parametric Differentiation Applied to the Falkner–Skan Equation

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The Falkner–Skan equation, which is the similarity form of the boundary layer equations, has been solved through the use of pseudo-spectral methods for different values of the similarity parameter β. The equation was then parametrically differentiated with respect to β and the resulting equation was solved with pseudo-spectral methods. A convergence criteria is established for each method. Comparisons are made between both approaches and the classical solutions.

INTRODUCTION

The Falkner–Skan equation

$$\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}+\beta\left[1-\left(\frac{df}{d\eta}\right)^2\right]=0, \qquad f(0)=f'(0)=0, \qquad f'(\infty)=1 \quad (1)$$

was chosen as the equation of interest because of the inherent nonlinearity which it exhibits. Additionally, since the solution was well known there was a large body of data with which to compare. Also, since the solution is continuous, the pseudo-spectral expansion will not exhibit Gibbs phenomena as is the case when discontinuities (shocks) are present. The equation is derived in Schlicting [1] and other textbooks on boundary layer theory.

Our interests in pseudo-spectral techniques are due to their potential to provide more efficient algorithms for solution of some nonlinear problems in fluid dynamics. The essence of pseudo-spectral techniques is that the solution can be represented as a series expansion or orthogonal functions. The particular functions chosen to be the basis functions depends largely on the properties of the solution one expects but functions which are smooth and continuous are generally chosen.

The method of parametric differentiation is a technique whereby a set of nonlinear differential equations is transformed into an equivalent set of linear differential

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equations coupled with a set of simple quadratures. The method of parametric differentiation has been shown to accelerate finite difference algorithms and to provide robust design schemes in nonlinear external flow fluid dynamics [2].

In this paper we will present a short discussion of the theory of both pseudo-spectral methods and of parametric differentiation as well as their application to the Falkner–Skan equation. The numerical techniques used will then be presented followed by a comparison of the calculations with the classical results. A section on conclusions and extensions will then follow.

Spectral Methods

While pseudo-spectral techniques have been in existence for many years, they have become increasingly popular over the past fifteen years. The motivating idea behind pseudo-spectral methods is the expansion of an unknown function in a series of orthogonal functions ϕ for which the strength of each term can be found. While the functions can be chosen arbitrarily, the choice can be important. The form of the expansion is: $y(x) = \sum_{n=0}^{\infty} A_n \phi_n(x)$. In the Galerkin approximation, for example, one chooses the functions so that the boundary conditions are satisfied identically. In a collocation pseudo-spectral expansion the error at selected points is formally set equal to zero. The third type of approximation, the tau approximation, truncates the series after N terms giving N equations for N coefficients. This leads to a situation where, if there are M boundary conditions, the last M of the N equations are replaced by the boundary conditions.

A major drawback, however, has been the strong sensitivity which pseudo-spectral methods have exhibited with respect to boundary conditions [3, 4]. If extraneous or slightly incorrect conditions are imposed on the problem the solution may not converge or convergence may be to an incorrect solution.

Parametric Differentiation

The method of parametric differentiation (MPD) [5] involves differentiating the governing equation and the boundary and/or initial conditions with respect to a parameter which may or may not appear explicitly in the formulation of the problem. The utility in this method is that nonlinear differential equations are transformed into *linear* equations with variable coefficients and the nonlinearity is transferred to the parameter integration.

The procedure is as follows [6, 7]: consider a set of possibly nonlinear equations

$$L_i(\phi_1, \phi_2, \phi_3, \ldots, \phi_n) = 0, \qquad i = 1, 2, \ldots, m \tag{2}$$

with the boundary conditions:

$$\text{at} \quad \mathbf{x}_i = \mathbf{x}_i^*, \qquad M_i(\phi_1, \phi_2, \phi_3, \ldots, \phi_n) = 0, \qquad i = 1, 2, \ldots, l, \tag{3}$$

where the L_i may be a general integro-differential operator and the ϕ_i are general functions of both the coordinates x_i and the parameter family λ. Assume that this set of equations has a known solution for some value of λ say λ_0. That is, $\phi_j = \phi_j^0(\mathbf{x}, \lambda_0)$. If (2) is now differentiated with respect to λ the resulting equation is

$$\hat{L}_i(\Phi_1, \Phi_2, \Phi_3, ..., \Phi_n) = 0, \qquad i = 1, 2, ..., m, \tag{4}$$

where $\hat{L}$ is a *linear* operator which operates on the Φ_j and

$$\Phi_j = \frac{\partial \phi_j}{\partial \lambda}, \qquad j = 1, 2, ..., k. \tag{5}$$

The boundary equations (3) are also differentiated to give

$$\text{at} \quad \mathbf{x}_i = \mathbf{x}_i^*, \qquad \hat{M}_i(\Phi_1, \Phi_2, \Phi_3, ..., \Phi_n) = 0, \qquad i = 1, 2, ..., n. \tag{6}$$

The original ϕ_j can be recovered by noting that

$$\phi_j(\mathbf{x}, \lambda) = \phi_j^0(\mathbf{x}, \lambda_0) + \int_{\lambda_0}^{\lambda} \Phi_j \, d\lambda. \tag{7}$$

The nonlinearity of the problem, therefore has been isolated into an integral which can be evaluated numerically.

By using MPD on the Falker–Skan equation (1) with the parameter taken to be β one has [2]:

$$\begin{aligned} g''' + fg'' + f''g - 2\beta f'g' &= f'^2 - 1, \\ g(0) = g'(0) = 0; \qquad g'(\infty) &= 0, \end{aligned} \tag{8}$$

where

$$g(\eta) = \frac{df}{d\beta}. \tag{9}$$

Consequently, the equations to be solved are (1) and (8). The numerical method used to solve them is the subject of the next section.

Numerics

For the numerical solution of the Falkner–Skan equation the basis function chosen was of the form $\phi = \{\cos nx\}$. However, it is not possible to use this basis function for f directly in terms of η since the boundary condition at infinity precludes any hope of a solution. Therefore, the infinite domain must be transformed into a region which is finite. A transformation which was found to be successful was: $\eta = \tan\theta$. The pseudo-spectral expansion was carried out in terms of θ. With this transformation the boundary conditions are imposed at $\theta=0$ and at $\theta=\pi/2-\varepsilon$. Where $\varepsilon=0.165$ was chosen that *infinity* was approximately 6 which is sufficiently far from the origin so that $u/U=1$. The boundary conditions were handled with a tau approximation except that the first and last equations (those for $\theta=0$, $\pi/2-\varepsilon$) were replaced by the appropriate boundary conditions. When Eq. (1) is evaluated at every point as required by collocation a nonlinear system of equation in terms of the A_n's results. The A_n's were found by using a packaged subroutine from the MIT Numerical Algorithms Group (NAG) library. While not the most efficient way to solve the problem, the actual programming was greatly simplified. The first $(n-2)$ collocation points were spaced as $\cos^{-1}(-\pi nx/N)$ and the remaining points were fixed at 0 and 6.

The solution of the MPDed Falkner–Skan equation proceeded in much the same way. In this case g was expanded in a cosine series. The base solution for f was taken to be the pseudo-spectral solution to the Blasius equation, i.e., the Falkner–Skan equation with $\beta=0$. The value for the base solution in g was found from the MPDed representation, again with β taken to be zero. New values for f were found by using Euler's method with $\Delta\beta=0.05$. Expressions for the derivatives of f were found assuming that f could be expanded in a cosine series and working backward to find the coefficients. That is, expressing f at $n+1$ nodes, we may write

$$\begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} \text{Matrix of} \\ \text{cosines} \end{bmatrix}^{-1} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{10}$$

Therefore, one can find f, f', f'', and f''' so that the coefficients of the MPDed Falkner–Skan equation are determined. The NAG routine was then used to find the coefficients in the expansion for g. The process is then repeated until the value of β_{max} is acheived. The results for this procedure are presented in the following section.

TABLE I

CPU Time											
Pseudo-spectral											
β	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Time (sec)	2	42	103	209	337	408	581	672	985	1142	1356
Pseudo-spectral/MPD											
β	0.05	0.1	0.15	0.20	0.25	0.30					
Total time for all cases (sec)						1539					

RESULTS

All calculations were performed on an IBM 370/168 operating under VM/CMS. CPU times varied from a few seconds to several minutes as indicated in Table I. No firm convergence criteria was established for the pseudo-spectral code because one is solving for the coefficients in the expansion. These coefficients are in turn multiplied by a number which is less than or equal to one in absolute value and then added together. Therefore, the convergence criteria does not determine an error bound on the series expansion but rather it establishes an error bound only on the coefficients of the expansion. The convergence criteria for the pseudo-spectral/MPD case is necessary to allow the computation to proceed to another value of the parameter β.

TABLE II

Coefficients of f

	β =-.1988	β = 0.0	β = 0.1	β = 0.2
A (0)	2.783189116D+04	2.5804943D+04	2.664205431D+04	2.645273835D+04
A (1)	-4.643907693D+04	-4.3197033D+04	-4.495026665D+04	-4.471464728D+04
A (2)	2.363224216D+04	2.2390489D+04	2.430188320D+04	2.441091894D+04
A (3)	1.538653257D+03	5.98772690D+02	-1.377802848D+03	-1.840703486D+03
A (4)	-1:879808560D+04	-1.6422506D+04	-1.460286891D+04	-1.394593283D+04
A (5)	2.425164573D+04	2.1535237D+04	2.008935747D+04	1.944801896D+04
A (6)	-2.057896193D+04	-1.8358534D+04	-1.738264689D+04	-1.690002583D+04
A (7)	1.320301297D+04	1.1870459D+04	1.131815863D+04	1.102890651D+04
A (8)	-6.755626205D+03	-6.0420801D+03	-5.784168758D+03	-5.645597234D+03
A (9)	2.706078482D+03	2.4218988D+03	2.324857643D+03	2.272560983D+03
A (10)	-8.334615556D+02	-7.4675552D+02	-7.184149697D+02	-7.033748518D+02
A (11)	1.872021122D+02	1.6807182D+02	1.620369085D+02	1.589368527D+02
A (12)	-2.749743871D+01	-2.4777383D+01	-2.394462663D+01	-2.353803394D+01
A (13)	1.995386577D+00	1.8088377D+00	1.753184549D+00	1.727904172D+00

	β = 0.3	β = 0.4	β = 0.5	β = 0.6
A (0)	2.717371710D+04	2.793079522D+04	2.912550357D+04	2.994828802D+04
A (1)	-4.630176633D+04	-4.767793001D+04	-4.997832936D+04	-5.155848121D+04
A (2)	2.597974688D+04	2.733856923D+04	2.938640536D+04	3.078280368D+04
A (3)	-3.411688133D+03	-4.572827433D+03	-6.246896015D+03	-7.376130849D+03
A (4)	-1.253521764D+04	-1.163353699D+04	-1.038969299D+04	-9.561360965D+03
A (5)	1.834710123D+04	1.772407142D+04	1.689448780D+04	1.634946220D+04
A (6)	-1.616660287D+04	-1.579109417D+04	-1.530143158D+04	-1.498403445D+04
A (7)	1.061772996D+04	1.042426672D+04	1.017260769D+04	1.001162340D+04
A (8)	-5.454820656D+03	-5.371571856D+03	-5.261133354D+03	-5.191428001D+03
A (9)	2.201005633D+03	2.171966560D+03	2.131612213D+03	2.106520954D+03
A (10)	-6.824433727D+02	-6.745754260D+02	-6.627263848D+02	-6.554994980D+02
A (11)	1.544360472D+02	1.528889355D+02	1.502473257D+02	1.486803827D+02
A (12)	-2.290168903D+01	-2.270680666D+01	-2.230521932D+01	-2.207682521D+01
A (13)	1.683155771D+00	1.671527802D+00	1.639914010D+00	1.623010190D+00

	β = 0.7	β = 0.8	β = 0.9	β = 1.0
A (0)	3.039547669D+04	3.141230768D+04	3.254792463D+04	3.317779892D+04
A (1)	-5.242267194D+04	-5.434703967D+04	-5.650867206D+04	-5.770771044D+04
A (2)	3.155912982D+04	3.318929183D+04	3.505140825D+04	3.608469579D+04
A (3)	-8.017052759D+03	-9.252416922D+03	-1.070083175D+04	-1.150547115D+04
A (4)	-9.082734806D+03	-8.246409664D+03	-7.233355956D+03	-6.669111836D+03
A (5)	1.603181441D+04	1.552709934D+04	1.489382555D+04	1.453935077D+04
A (6)	-1.480015800D+04	-1.452956713D+04	-1.417871575D+04	-1.398066867D+04
A (7)	9.920680677D+03	9.792551244D+03	9.622232779D+03	9.524838649D+03
A (8)	-5.153938264D+03	-5.100892326D+03	-5.029576947D+03	-4.988032139D+03
A (9)	2.094053757D+03	2.075193417D+03	2.050006443D+03	2.034961671D+03
A (10)	-6.523099976D+02	-6.467320953D+02	-6.394791661D+02	-6.350061774D+02
A (11)	1.480989085D+02	1.468044129D+02	1.451950481D+02	1.441634976D+02
A (12)	-2.201129616D+01	-2.180113911D+01	-2.155435902D+01	-2.138905822D+01
A (13)	1.619765136D+00	1.601855267D+00	1.582050515D+00	1.568134920D+00

TABLE III

Coefficients of g

	β = 0.0	β = 0.05,0.10,0.15	β = 0.2
A (0)	-0.84263774D+05	-0.53611007D+05	-0.42439397D+05
A (1)	0.15238996D+06	0.98610311D+05	0.77578746D+05
A (2)	-0.11105539D+06	-0.76269815D+05	-0.58745040D+05
A (3)	0.61359681D+05	0.48603196D+05	0.35712703D+05
A (4)	-0.20085992D+05	-0.24234996D+05	-0.15906053D+05
A (5)	-0.41095808D+04	0.80467466D+04	0.33544137D+04
A (6)	0.12235031D+05	-0.22769236D+03	-0.20529487D+04
A (7)	-0.10805952D+05	-0.19508993D+04	-0.28930523D+04
A (8)	0.64342424D+04	0.16248978D+04	0.19489045D+04
A (9)	-0.28500766D+04	-0.81955297D+03	-0.90944529D+03
A (10)	0.94468153D+03	0.28988068D+03	0.30910371D+03
A (11)	-0.22493797D+03	-0.71579023D+02	-0.74513974D+02
A (12)	0.34705571D+02	0.11290756D+02	0.11567831D+02
A (13)	-0.26297027D+01	-0.86906419D+00	-0.88035075D+00

	β = 0.25	β = 0.3
A (0)	0.26706226D+05	-0.8765503553D+05
A (1)	-0.52499435D+05	0.1619500996D+06
A (2)	0.49402757D+05	-0.1271298690D+06
A (3)	-0.43529115D+05	0.8355753783D+05
A (4)	0.34999921D+05	-0.4444586331D+05
A (5)	-0.25093522D+05	0.1754939209D+05
A (6)	0.15726928D+05	-0.3586681517D+04
A (7)	-0.84565482D+04	-0.1277369867D+04
A (8)	0.38215320D+04	0.1734563782D+04
A (9)	-0.14134976D+04	-0.9850701397D+03
A (10)	0.41198092D+03	0.3683583388D+03
A (11)	-0.89036113D+02	-0.9406959266D+02
A (12)	0.12740847D+02	0.1519008989D+02
A (13)	-0.91067694D+00	-0.1189784020D+01

The criteria was that the sum of the squares of the residuals of the coefficients had to be less than or equal to 0.2. By increasing this number, convergence could be attained more rapidly with probably little degradation in the series solution.

For each case, the initial coefficients were obtained from the previous case. So that for $\beta = 0.4$ the solution for $\beta = 0.3$ was input as an initial guess. For $\beta = 0.0$ the coefficients were estimated from knowing the solution for a flat plate (Blasius solution).

The results of the calculations from both the pseudo-spectral and the spectral/MPD codes are encouraging. The coefficients for the pseudo-spectral expansion are listed in Tables II and III. Note that the coefficients decrease so that the higher modes contribute less than the lower modes as one would expect when one considers that terms like $A_0 \cos\theta$ and $A_1 \cos\theta$ will define the gross behavior of the curve.

Figures 1 and 2 were obtained from the pseudo-spectral code. Figure 3 was obtained from the pseudo-spectral/MPD code and Fig. 4 presents a comparison between both calculations. The errors in the MPD code mainly result from the parameter integration since it is only first order accurate. If a higher order method, e.g., a predictor–corrector method had been used the errors would have been substantially less.

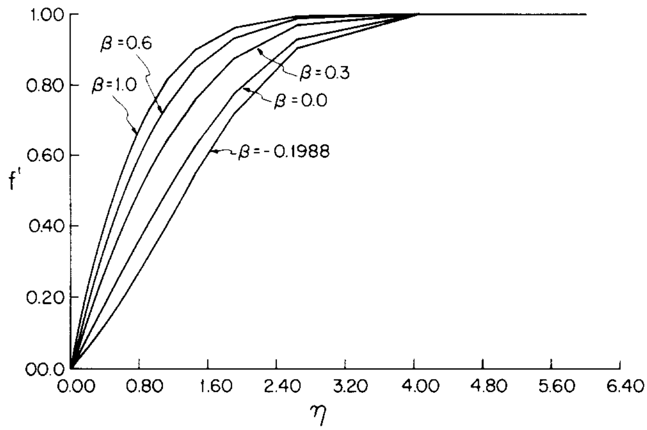

FIG. 1. Pseudo-spectral solution of the Falkner–Skan equation for different values of β.

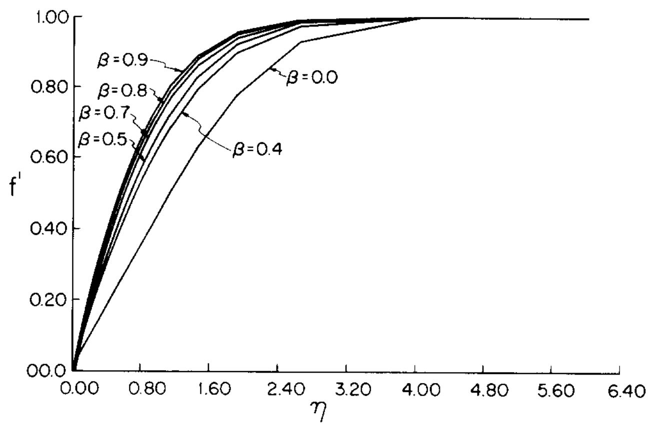

FIG. 2. Pseudo-spectral solution of Falkner–Skan equation for different values of β.

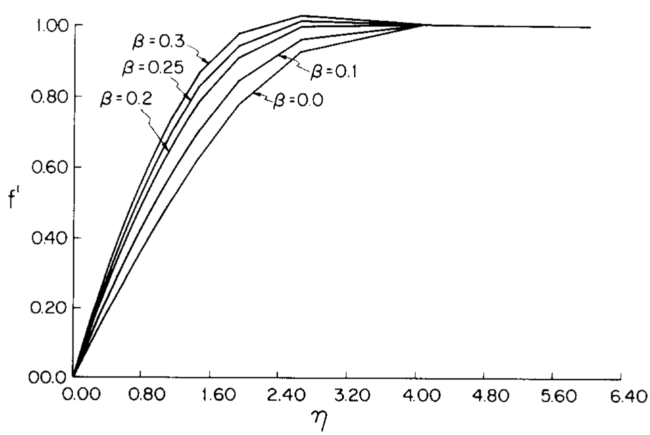

FIG. 3. Pseudo-spectral/MPD solution of the Falkner–Skan equation for different values of β.

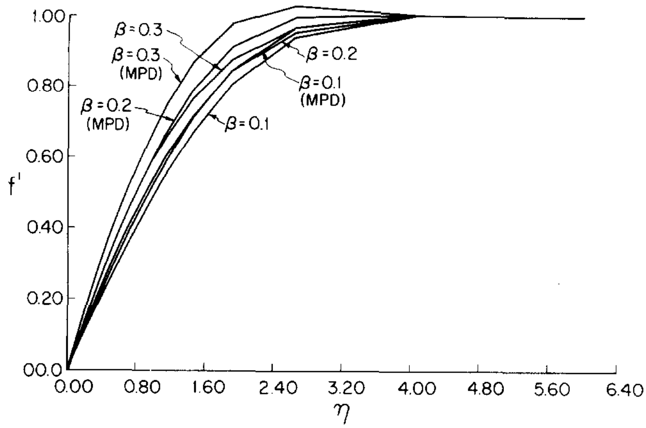

FIG. 4. Comparison between pseudo-spectral and pseudo-spectral/MPD results.

CONCLUSIONS

From studying Table I and Figs. 1–7 we arrive at the following conclusions:

(a) Pseudo-spectral techniques can be successfully applied not only to linear differential equations [8] but also to both differential equations with variable coefficients and to nonlinear differential equations.

(b) Comparison of pseudo-spectral technique with the classical values presented by Rosenhead shows agreement to within ten percent for $\beta = 0$ and to within two percent for $\beta = 1.0$.

(c) Comparison of results obtained by using pseudo-spectral techniques and the method of parametric differentiation with results obtained by pseudo-spectral techniques alone show favorable agreement.

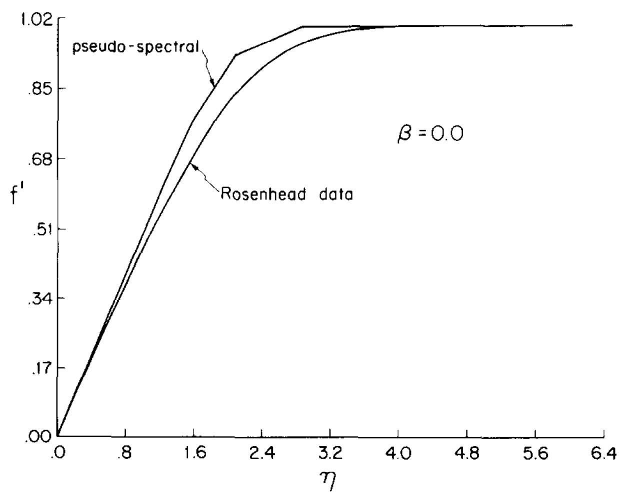

FIG. 5. Comparison of data from Rosenhead with that from the pseudo-spectral code for $\beta = 0.0$.

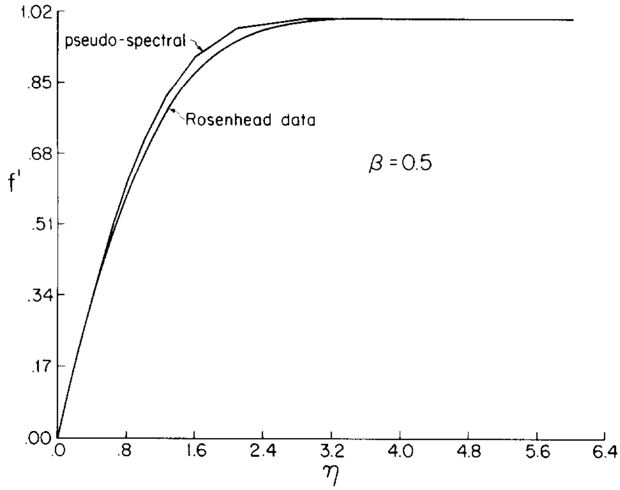

FIG. 6. Comparison of data from Rosenhead with that from the pseudo-spectral code for $\beta = 0.5$.

While the results reported in this paper clearly establish that pseudo-spectral techniques may be successfully applied to nonlinear differential equations, several recommendations are offered as a means of enhancing the efficiency of the computer codes developed.

(a) Because of the way coefficients are found, namely, through the use of a packaged nonlinear equation solver, the CPU times for the problem under consideration are rather large. If the finding of the coefficients had been optimized, the CPU times would undoubtedly decrease dramatically. In fact, for complicated

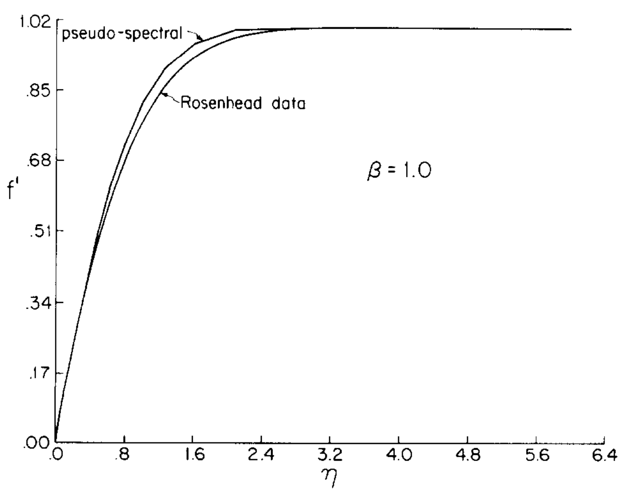

FIG. 7. Comparison of data from Rosenhead with that from the pseudo-spectral code for $\beta = 1.0$.

problems such as incompressible turbulence modeling for free shear flows [9, 10] or the transonic small disturbance equation [11] pseudo-spectral methods have been shown to be at least comparable to, if not superior to finite difference methods. In the solution of the Falkner–Skan equation the majority of the CPU time appears to have been used in the packaged routine to solve for the coefficients.

(b) A more accurate parameter integration would, undoubtedly, produce results which are of comparable accuracy to non-MPDed calculations but with a shorter computation time [13].

(c) If solutions are required over a sufficiently large interval of values of the parameter of interest, a coupling of the method of parametric differentiation and pseudo-spectral methods could be more efficient than using a pseudo-spectral code alone.

(d) Different orthogonal functions, Chebyshev polynomials, for example, could be chosen for the expansion. However, since derivatives of the fuction are needed, choosing a basis with easily found derivatives would be useful.

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